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Electro-Magnetic Space-Time Duality for $2+1$ D Stationary Classical Solutions

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Abstract

We extend a space-time duality suggested by Kogan for $2+1$-dimensional static classical solutions of Einstein Maxwell Chern-Simons theories to stationary rotating space-times. We also extend the original duality to other possible dualities of the same kind that constitute a close duality web. In $3+1$-dimensions these dualities are only possible for systems which exhibit non-projected cylindrical symmetry and are not related to the usual electromagnetic duality of Maxwell equations. Generalization to N-form theories in higher dimensional space-times is briefly addressed.

1 Introduction

It can be find in the literature gauge fields dualities that map electric into magnetic classical solutions which are reminiscent to the original electro-magnetic duality of Maxwell equations in four dimensional space-time [1,2]. Namely in the works of Deser and al. [3–5] such dualities are applied to black hole solutions in several dimensions.

Here we explore a space-time duality put forward by Kogan in [6,7] concerning $2+1$-dimensional Einstein Maxwell Chern-Simons theories for static radial symmetric metrics. This duality is a space-time duality instead of a field duality. We extend this duality to stationary rotating spaces and discuss two possible ways of implementing the duality that depend on the signs of the ADM metric. We also relate both dualities trough a third duality consisting of a double Wick rotation such that we obtain a close web of dualities. Briefly we discuss the meaning of these dualities in $3+1$-dimensional space-time as well as the application of the dualities to N-form theories in higher dimensional space-times.

In section 2 we review the original duality as put forward by Kogan and generalize it to rotating spaces. In section 3 we propose a new duality of the same kind and in section 4 we relate both dualities by a third duality which is a double Wick rotation such that we obtain a close web of dualities. Finally in section 5 we generalize the dualities to non-projected four dimensional solutions with cylindrical symmetry and briefly discuss N-form theories in higher dimensional space-times.

2 Original Duality for Rotating Space-Times

The original duality in [6,7] consist in transform electric into magnetic solutions (and vice-versa) by exchanging the role of the time coordinate t and the angular coordinate φ. So let us take a metric on the ADM form [8]

$$ds^2 = -\tilde{f}^2 dt^2 + dr^2 + \tilde{h}^2 (d\varphi + \tilde{A} dt)^2 \tag{2.1}$$

and the electric and magnetic field definitions

$$\begin{aligned} \tilde{E} &= \tilde{F}_{tr} \\ \tilde{B} &= \tilde{F}_{r\varphi} \ . \end{aligned} \tag{2.2}$$

Swapping directly the two variables corresponds to have the original transformation [6]

$$\begin{aligned} t &\to i\varphi \\ \varphi &\to it \ . \end{aligned} \tag{2.3}$$

In order to generalize this duality to rotating space-times we consider the Cartan triad $e^0 = f\,dt$ and $e^2 = h(d\varphi + A dt)$ (see [9]). Then the duality simple exchanges $e^0 \leftrightarrow e^2$

(see [12]). In terms of the several gravitational fields this accounts for the following duality

$$\begin{aligned} \tilde{f} &\to ih \\ \tilde{h} &\to if \\ \tilde{E} &\to -iB \\ \tilde{B} &\to -iE \end{aligned} \tag{2.4}$$

without changing the remaining fields. Then we obtain the metric

$$ds^2_{\text{dual}} = -f^2(dt + Ad\varphi)^2 + dr^2 + h^2 d\varphi^2 \tag{2.5}$$

As for the gauge sector the Maxwell and Chern-Simons terms transform as $\tilde{F}\wedge*\tilde{F}+\tilde{A}\wedge\tilde{F} \to -F\wedge*F-A\wedge F$. So the gauge sector swaps the relative sign with respect to the gravitational sector.

We note that it is quite interesting that the duality is a transformation on space-time coordinates, although the gauge fields transform accordingly this is not a duality of the gauge fields.

The usual components of the two metrics corresponding to ds^2 and ds^2_{dual} are

$$\begin{array}{rcl} \tilde{g}_{00} &=& -\tilde{f}^2 + \tilde{h}^2\tilde{A}^2 \\ \tilde{g}_{11} &=& 1 \\ \tilde{g}_{22} &=& \tilde{h}^2 \\ \tilde{g}_{02} &=& \tilde{h}^2\tilde{A} \end{array} \qquad \text{and} \qquad \begin{array}{rcl} g_{00} &=& -f^2 \\ g_{11} &=& 1 \\ g_{22} &=& h^2 - f^2A^2 \\ g_{02} &=& -f^2A\ . \end{array} \tag{2.6}$$

We can relate this two parameterization by the following map

$$\begin{aligned} \tilde{f}^2 &= \frac{f^2\,h^2}{h^2 - f^2\,A^2} \\ \tilde{h}^2 &= h^2 - f^2\,A^2 \\ \tilde{A} &= -\frac{A\,f^2}{h^2 - f^2\,A^2} \end{aligned} \tag{2.7}$$

As for the ADM metric signature we obtain the following cases

$$\begin{aligned} h^2 - f^2\,A^2 > 0 &\;\Rightarrow\; \text{Map maintains ADM metric signature} \\ h^2 - f^2\,A^2 < 0 &\;\Rightarrow\; \text{Map changes ADM metric signature}\ . \end{aligned} \tag{2.8}$$

This is directly seen by rewriting the metric in the ADM form (2.1) by using the above map (2.7).

So we conclude that the duality (2.4) not always gives the same metric signature, only for $h^2 - f^2\,A^2 > 0$ we obtain the same Minkowski signature. We note that this problem only arise in rotating spaces, by putting $A \to 0$ we get that the $h^2 > 0$ (as long as h is real) and the metric always maintains the signature under this duality.

3 Another Possible Duality for Rotating Space-Times

For the cases where we have $h^2 - f^2\,A^2 < 0$ there is yet another way of implementing our duality such that the ADM form of the metric hold the correct Minkowski signature of the metric. Let us consider simply the swapping of the time and angular coordinates

$$\begin{aligned} t &\rightarrow \varphi \\ \varphi &\rightarrow t\,. \end{aligned} \tag{3.1}$$

In terms of the several fields this accounts for the following duality

$$\begin{aligned} \tilde{f} &\rightarrow \hat{h} \\ \tilde{h} &\rightarrow \hat{f} \\ \tilde{E} &\rightarrow -\hat{B} \\ \tilde{B} &\rightarrow -\hat{E} \end{aligned} \tag{3.2}$$

without changing the remaining fields. Here we use hated fields to distinguish between the two dualities as given in (2.4) and (3.2). We obtain the metric

$$d\hat{s}^2_{\rm dual} = \hat{f}^2(dt + \hat{A}d\varphi)^2 + dr^2 - \hat{h}^2 d\varphi^2 \tag{3.3}$$

As for the gauge sector we note that the Maxwell and Chern-Simons terms transform as $\tilde{F}\wedge *\tilde{F} + \tilde{A}\wedge\tilde{F} \rightarrow +\hat{F}\wedge *\hat{F} + \hat{A}\wedge\hat{F}$. So the gauge sector maintains the relative sign with respect to the gravitational sector.

The usual components of the dual metric (3.3) are

$$\begin{aligned} \hat{g}_{00} &= \hat{f}^2 \\ \hat{g}_{11} &= 1 \\ \hat{g}_{22} &= -\hat{h}^2 + \hat{f}^2\hat{A}^2 \\ \hat{g}_{02} &= \hat{f}^2\hat{A}\,. \end{aligned} \tag{3.4}$$

We can relate these two parameterization by the following map

$$\begin{aligned} \tilde{f}^2 &= \frac{\hat{f}^2\,\hat{h}^2}{-\hat{h}^2 + \hat{f}^2\,\hat{A}^2} \\ \tilde{h}^2 &= -\hat{h}^2 + \hat{f}^2\,\hat{A}^2 \\ \tilde{A} &= -\frac{\hat{A}\,\hat{f}^2}{-\hat{h}^2 + \hat{f}^2\,\hat{A}^2} \end{aligned} \tag{3.5}$$

Concerning the behaviour of the metric signature under the above map we have the following cases:

$$\begin{aligned} -\hat{h}^2 + \hat{f}^2\,\hat{A}^2 > 0 &\;\Rightarrow\; \text{Map maintains ADM metric signature} \\ -\hat{h}^2 + \hat{f}^2\,\hat{A}^2 < 0 &\;\Rightarrow\; \text{Map changes ADM metric signature}\,. \end{aligned} \tag{3.6}$$

4 Double Wick Rotation as a Duality

Both dualities as given by (2.4) and (3.2) are related by a double Wick rotation

$$\begin{aligned} t &\rightarrow it \\ \varphi &\rightarrow i\varphi \end{aligned} \tag{4.7}$$

such that the fields transform accordingly has

$$\begin{aligned} f &\rightarrow i\hat{f} \\ h &\rightarrow i\hat{h} \\ E &\rightarrow i\hat{E} \\ B &\rightarrow i\hat{B} \end{aligned} \tag{4.8}$$

such that the factor $-h^2 + f^2\, A^2 \rightarrow \hat{h}^2 - \hat{f}^2\, \hat{A}^2$ swaps sign. The double Wick rotation also affects the gauge sector $F \wedge *F + A \wedge F \rightarrow -\hat{F} \wedge *\hat{F} - \hat{A} \wedge \hat{F}$ such that the relative sign between the gauge and gravitational sector is swapped.

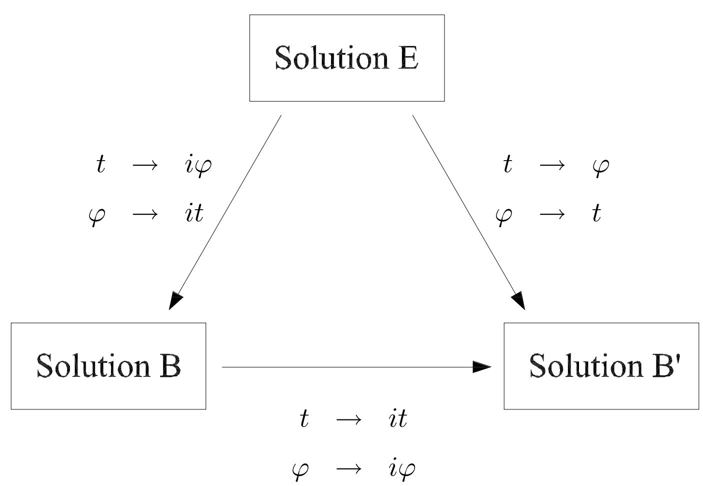

Figure 1: *Web of Dualities. The dualities can be considered in both directions. In the picture are expressed only the dualities for the directions of the arrows.*

In this way from a given electric or magnetic solution for the metric parameterization (2.1) one can find a magnetic or electric solution (respectively) using the above dualities (2.4) and (3.2). The choice of the duality to be employed depends on the specific form of the solutions such that the dual metric signature is set accordingly either by condition (2.8) or (3.6). Also one can from a given magnetic solution with a given ADM metric signature obtain another magnetic solution with a different ADM metric signature. This web of dualities is pictured in the following figure 1.

5 Dualities in $3+1$D and Higher Dimensional Space-Times

In $2+1$-dimensional the magnetic field is a scalar (corresponding to $\tilde{B}_{(3D)} = \tilde{F}_{r\varphi}$). Usually a planar system can be interpreted as a projected $3+1$-dimensional system. So far our dualities apply to such frameworks and we have maintained the angular component of the electric field null, $\tilde{E}^{\varphi}_{(3D)} = \tilde{F}^{0\varphi} = 0$. Due to the Einstein equations, if such component of the electric field is not null, we will need to further consider a radial shift function in the metric [12] which does not constitute a physical gravitational degree of freedom in $2+1$-dimensional gravity [9].

More generally in $3+1$-dimensional we can assume non-projected cylindrical symmetry around the θ direction such that the projected magnetic field corresponds to the θ component of the non-projected magnetic field, $\tilde{B}_{(3D)} = \tilde{B}^{\theta}_{(4D)}$. Concerning the projection of four dimensional gravity in three dimensional space-times we refer the reader to [10, 11].

Then for four dimensional solutions with cylindrical symmetry we obtain two possible components for the electric and magnetic field

$$\left\{ \begin{array}{lcl} \tilde{E}^{r}_{(4D)} & = & \tilde{F}^{0r} \\ \\ \tilde{E}^{\theta}_{(4D)} & = & \tilde{F}^{0\theta} \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{lcl} \tilde{B}^{r}_{(4D)} & = & \tilde{F}_{\varphi\theta} \\ \\ \tilde{B}^{\theta}_{(4D)} & = & \tilde{F}_{r\varphi} \end{array} \right. . \tag{5.1}$$

such our dualities exchange the radial and polar electric fields with the polar and radial magnetic fields respectively. The first duality (2.4) reads

$$\begin{array}{rcl} \tilde{E}^{r}_{(4D)} & \to & -iB^{\theta}_{(4D)} \\ \\ \tilde{E}^{\theta}_{(4D)} & \to & -iB^{r}_{(4D)} \\ \\ \tilde{B}^{r}_{(4D)} & \to & -iE^{\theta}_{(4D)} \\ \\ \tilde{B}^{\theta}_{(4D)} & \to & -iE^{r}_{(4D)} \end{array} \tag{5.2}$$

and the second duality (3.2) reads

$$\begin{array}{rcl} \tilde{E}^{r}_{(4D)} & \to & -\hat{B}^{\theta}_{(4D)} \\ \\ \tilde{E}^{\theta}_{(4D)} & \to & -\hat{B}^{r}_{(4D)} \\ \\ \tilde{B}^{r}_{(4D)} & \to & -\hat{E}^{\theta}_{(4D)} \\ \\ \tilde{B}^{\theta}_{(4D)} & \to & -\hat{E}^{r}_{(4D)} \end{array} \tag{5.3}$$

In this way we lifted the duality from a planar system to a $3+1$-dimensional system with cylindrical symmetry. As expected these sort of dualities are not possible for generic stationary solutions in four dimensions that do not possess cylindrical symmetry. In order

to conclude it is enough to consider the remaining field components $\tilde{E}^{\varphi}_{(4D)} = \tilde{F}^{0\varphi}$ and $\tilde{B}^{\varphi}_{(4D)} = -\tilde{F}_{r\theta}$ which are maintained (up to sign changes) by the dualities (2.4) and (3.2).

It is interesting to note that a generalization is possible for N-form theories in higher dimensional space-times [5]. For instance in $5+1$-dimensions, by considering the 2-form fields $E^{IJ} = F^{0IJ}$ and $B^{IJ} = \epsilon^{IJKLM} F_{KLM}/6$, under the duality $(t \to ix^5,\ x^5 \to it)$ we obtain the map $E^{12} \leftrightarrow -iB^{34}$, $E^{13} \leftrightarrow -iB^{24}$, $E^{14} \leftrightarrow -iB^{23}$, $E^{23} \leftrightarrow -iB^{14}$, $E^{24} \leftrightarrow -iB^{13}$ and $E^{34} \leftrightarrow -iB^{12}$.

We note that there is no relation of these dualities with the usual electromagnetic duality of the gauge fields [1–5]. The original electromagnetic duality rotates the same components of the electric and magnetic into each other not mixing the space-time components of the fields.

6 Conclusions

We built a space-time duality web that allows to map electric into magnetic classical solutions and vice-versa. These dualities may generally change the ADM metric signature as given in (2.1). Then, given that the coefficient of dr^2 is positive (see [12] for a discussion on different choices of signatures), the original Minkowski signature is considered to be the one that holds the coefficient of dt^2 negative and the coefficient of $d\varphi^2$ positive, this means that we are aiming at solutions with $\tilde{f}^2 > 0$ and $\tilde{h}^2 > 0$. Also from the explicit form of the three possible dualities (2.4), (3.2) and (4.8) we may map real solutions into imaginary solutions. Nevertheless by imposing reality conditions and properly constraining the parameters and variables of our the original solutions we are able to achieve only real solutions.

There is a very interesting feature of the dualities presented in this work. It is possible from a particle solution to obtain a instanton solution and vice-versa by means of a double Wick rotation (4.8). We will study this feature in a forthcoming work [13].

We also concluded that such dualities are not related to the usual electromagnetic duality of the gauge fields and although valid for four dimensional solutions with cylindrical symmetry cannot be generalized to generic stationary solutions in four dimensions. However we give an example in six dimensional space-time that show that our dualities can be generalized to N-form theories in higher dimensional space-times.

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